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8. CLR bound

9. LT inequality

8. The CLR bound

The Cwikel – Lieb – Rozenblum inequality provides a bound on the number of negative eigenvalues of a Schrödinger operator: $N(0, -\Delta+V)$.

Thm (Cwikel-Lieb-Rozenblum inequality)
Let $d \geq 3$ and $V_- \in L^{d/2}(\mathbb{R}^d)$. Then

$$N(0, -\Delta+V) \leq C_d \int_{\mathbb{R}^d} |V_-|^{d/2}$$

where C_d depends only on the dimension.

Remark
It is not surprising that the positive part of V does not appear in the upper bound because $N(0, -\Delta+V) \leq N(0, -\Delta+V_-)$ by the min-max principle.

Proof (due to Rupert Frank)
Let W be the space spanned by eigenfunctions of negative eigenvalues of $-\Delta+V$. Assume $\dim W \geq N$. Since the operator $(-\Delta)^{1/2}$ is strictly positive on $L^2(\mathbb{R}^d)$ we have

$$\dim((-\Delta)^{1/2} W) \geq N. \quad (*)$$

Indeed: $\sqrt{-\Delta}$ is positive a square root of a positive operator. Furthermore, if $\{u_1, \dots, u_n\}$

are linearly independent, then so are $\{\sqrt{-\Delta}\, u_i\}$.

To see this, note that

$$\sqrt{-\Delta}\, u_i - t\sqrt{-\Delta}\, u_j = 0 \iff$$

$$\sqrt{-\Delta}\,(u_i - t u_j) = 0 \underset{\substack{\uparrow \\ \sqrt{-\Delta} > 0 \text{ so inverse exists}}}{\iff} u_i - t u_j = 0$$

This justifies (*).

Thus we can choose $\{u_n\}_{n=1}^N \subset W$ such that $\{\sqrt{-\Delta}\, u_n\}_{n=1}^N$ are orthonormal in $L^2(\mathbb{R}^d)$.

(indeed: first choose $\tilde{u}_n$ ON (Gram-Schmidt)

$\rightsquigarrow$ $u_n := \sqrt{-\Delta}^{-1}\tilde{u}_n$).

We now use the kinetic LT inequality (see part 2):

Let $d > 2\kappa \geq 0$ and $s \geq 0$ (κ, s are not necessarily integers). For any $N \geq 1$, let $\{(-\Delta)^{\kappa/2} u_n\}_{n=1}^N$ be orthonormal functions in $L^2(\mathbb{R}^d)$ and denote $\rho(x) = \sum_{n=1}^N |u_n(x)|^2$.

Then

$$\boxed{\sum_{n=1}^N \|(-\Delta)^{s/2} u_n\|^2_{L^2(\mathbb{R}^d)} \geq K_{d,s,\kappa} \int_{\mathbb{R}^d} \rho(x)^{1+\frac{2s}{d-2\kappa}}\, dx.}$$

The constant $K_{d,s,\kappa}$ is independent of N and $\{u_n\}$

We will use it for $\kappa = s = 1$. We have:

$$N = \sum_{n=1}^{N} \| (-\Delta)^{1/2} u_n \|^2_{L^2(\mathbb{R}^d)} \geq K_d \int_{\mathbb{R}^d} \rho(x)^{\frac{d}{d-2}} \, dx$$

On the other hand, since $\{u_n\} \subset W$ we have

$$0 \geq \sum_{n=1}^{N} \langle u_n, (\sqrt{-\Delta} + V) u_n \rangle_{L^2(\mathbb{R}^d)} = N + \int_{\mathbb{R}^d} V(x) \rho(x) \, dx$$

Putting together these inequalities we find that

$$N \leq - \int_{\mathbb{R}^d} V(x) \rho(x) \, dx \leq \int_{\mathbb{R}^d} |V_-(x)| \rho(x) \, dx$$

$$\overset{\text{Hölder}}{\leq} \|V_-\|_{L^{\frac{d}{2}}} \|\rho\|_{L^{\frac{d}{d-2}}} \leq \|V_-\|_{L^{d/2}} \left(\frac{N}{K_d} \right)^{\frac{d-2}{d}}$$

which is equivalent to

$$N^{1 - 1 + \frac{2}{d}} \leq \|V_-\|_{\frac{d}{2}} K_d^{\frac{2}{d} - 1}$$

$$\Updownarrow$$

$$N \leq K_d^{1 - \frac{d}{2}} \int_{\mathbb{R}^d} |V_-(x)|^{\frac{d}{2}} dx$$

We conclude by taking $N \to \dim W$. $\square$

<u>Remark</u> The CLR bound fails in $d \leq 2$.

Exercise

Let $\int_{\mathbb{R}^d} V(x) < 0$ in $d = 1, 2$. Then $E_0 < 0$.

Solution:

Let us consider $\boxed{d=1}$. We take the trial state

$$\varphi_\lambda = \sqrt{\lambda}\, e^{-\lambda x^2}$$

Then

$$T_{\varphi_\lambda} = \int_{\mathbb{R}} \left| 2\lambda^{3/2} x\, e^{-\lambda x^2} \right|^2 dx =$$

$$= \int_{\mathbb{R}} 4\lambda^3 x^2 e^{-2\lambda x^2} dx = \left\| \begin{array}{l} \lambda x^2 = s^2 \\ \sqrt{\lambda} x = s \\ dx = \frac{ds}{\sqrt{\lambda}} \end{array} \right\| = \int_{\mathbb{R}} 4\lambda^3 \frac{s^2}{\lambda} e^{-2s} \frac{ds}{\sqrt{\lambda}}$$

$$= 4\lambda^{3/2} \int_{\mathbb{R}} s^2 e^{-2s} ds = c\lambda^{3/2}$$

$$V_{\varphi_\lambda} = -\int \lambda e^{-2\lambda x^2} V_-(x) \sim -\lambda c \quad (\text{as } \lambda \to 0)$$

$$T_{\varphi_\lambda} + V_{\varphi_\lambda} < 0 \quad \text{as } \lambda \to 0$$

•) $d=2$.

trial function $e^{-\lambda r^\alpha}$

$$E_0 \leq \frac{N}{D}, \text{ where } N = T+V$$

$$T_{\alpha,\lambda} = \int_0^\infty dr\, r \left(\frac{d}{dr} e^{-\lambda r^\alpha}\right)^2 ; \quad V_{\lambda,\alpha} = \int_0^\infty dr\, V e^{-2\lambda r^\alpha}$$

$$D_{\alpha,\lambda} = \int dr\, r\, e^{-2\lambda r^\alpha} \quad - \text{normalization}$$

$$\frac{V_{\lambda,\alpha}}{D_{\lambda,\alpha}} \longrightarrow \int V .$$

$$T_{\lambda,\alpha} = \int_0^\infty dr\, r \left(-\alpha\lambda r^{\alpha-1} e^{-\lambda r^\alpha}\right)^2 =$$

$$= \int_0^\infty dr\, r\, \alpha^2 \lambda^2 r^{2\alpha-2} e^{-2\lambda r^\alpha} = \left|\left| \begin{array}{l} \lambda r^\alpha = u \\ \alpha\lambda r^{\alpha-1} dr = du \\ u\, du = \alpha\lambda^2 r^{2\alpha-1} \end{array} \right.\right.$$

$$= \alpha \int_0^\infty u e^{-2u} du \Rightarrow \frac{T}{P} \to c\alpha$$

$$\Rightarrow E_0 < 0 \text{ for } \lambda \to 0, \alpha \to 0.$$

$\square$

There exist CLR type bounds in $d=1,2$ but they are slightly more complicated.

The CLR bound above can be also extended to the case of fractional Laplacians. To this end one repeats the proof with $k=s$ (in the kinetic inequality) to obtain

$$N \leq K^{1-\frac{d}{2s}} \int_{\mathbb{R}^d} |V_-(x)|^{\frac{d}{2s}}$$

$$\Rightarrow \quad \mathcal{N}\left((-\Delta)^s + V\right) \leq C_{d,s} \int_{\mathbb{R}^d} |V_-(x)|^{\frac{d}{2s}} dx.$$

We will now be interested in the sum of negative eigenvalues (not just the number).

9. Lieb - Thirring inequalities

Thm Let $d > 2s$ and $k > 0$, $V_- \in L^{k+\frac{d}{2s}}(\mathbb{R}^d)$. Let $\{E_n((-\Delta)^s+V)\}_{n\geq 1}$ denote the negative eigenvalues of $(-\Delta)^s+V$. Then

$$\sum_{n\geq 1} |E_n((-\Delta)^s+V)|^k \leq C_{k,d,s} \int_{\mathbb{R}^d} |V_-(x)|^{k+\frac{d}{2s}} dx$$

Proof

This result follows from the $k=0$ case proven above. We use the layer-cake representation

$$|E_n|^k = k \int_0^\infty \mathbb{1}(E_n < -E)\, E^{k-1}\, dE$$

Using the CLR bound for the negative eigenvalues of $(-\Delta)^s + V + E$ we get

$$\sum_{n=1} \mathbb{1}(E_n < E) \leq C \int_{\mathbb{R}^d} |(V(x)+E)_-|^{\frac{d}{2s}} dx$$

Thus

$$\sum_{n\geq 1} |E_n|^k = k \int_0^\infty \sum_{n\geq 1} \mathbb{1}(E_n < -E)\, E^{k-1}\, dE$$

$$\leq Ck \int_0^\infty \left(\int_{\mathbb{R}^d} |(V(x)+E)_-|^{\frac{d}{2s}} E^{k-1}\, dx \right) dE$$

$$= Ck \int_{\mathbb{R}^d} \left(\int_0^\infty |(V(x)+E)_-|^{\frac{d}{2s}} E^{k-1}\, dE \right) dx$$

But

$$\int_0^\infty |(V(x)+E)_-|^{\frac{d}{2s}} E^{k-1}\, dE =$$

$$= \int_0^\infty |V_-|^{\frac{d}{2s}} \left|\left(1+\frac{E}{|V_-|}\right)_-\right|^{\frac{d}{2s}} \left(\frac{E}{|V_-|}\right)^{k-1} |V_-|^{k-1}\, d\left(\frac{E}{|V_-|}\right)\cdot |V_-| =$$

$$= |V_-|^{\frac{d}{2s}+k} \int_0^\infty |(1-\tilde{E})_-|^{d/2s}\, \tilde{E}^{k-1}\, d\tilde{E}$$

Thus

$$\sum_{n\geq} |E_n|^k \leq C' \int_{\mathbb{R}^d} |V_-(x)|^{k+\frac{d}{2s}}\, dx$$

where $C' = Ck \int_0^\infty |(1-E)_-|^{d/2s} E^{k-1}\, dE$ $\boxed{\equiv}$

Remarks

-) the original LT inequality has been derived by Lieb and Thirring in 1975 for $k=1$, $s=1$
-) They extended the inequality to $k>0$ when $d\geq 2$ and $k>\frac{1}{2}$ when $d=1$ (all for $s=1$)
-) CLR bound in 1977. $k=\frac{1}{2}$ $d=1$ $\to$ Weidl 1996

We shall now sketch the original Lieb-Thirring proof.

Original Lieb-Thirring proof $(s>1, d=3)$

Let $-e$ be a negative eigenvalue of $H = -\Delta - V_-$ with eigenfct ψ. If

$$\phi(x) := \sqrt{V_-(x)}\,\psi$$

then

$$(-\Delta + e)\psi = \sqrt{V_-}\,\phi.$$

Lemma For a given f, the solution of $(-\Delta+e)g = f$ is given by

$$(-\Delta+e)^{-1} f := \int_{\mathbb{R}^3} G_e(x-y)\, f(y)\, dy$$

with

$$G_e(x-y) = \int_{\mathbb{R}^3} \frac{1}{|2\pi k|^2 + e}\, e^{2\pi i k\cdot(x-y)}\, dk$$

Using this we have

$$\phi = \underbrace{\sqrt{V_-}\,(-\Delta+e)^{-1}\sqrt{V_-}}_{K_e}\,\phi$$

which can be rewritten as $K_e \phi = \phi$

K_e is an integral operator, i.e.

$$(K_e\phi)(x) = \int_{\mathbb{R}^3} K_e(x,y)\,\phi(y)\,dy$$

with

$$K_e(x,y) = \sqrt{V_-(x)}\; G_e(x-y)\,\sqrt{V_-(y)}$$

which is called the Birman-Schwinger kernel.

Thm (Birman - Schwinger principle)

1. K_e is a bounded operator on $L^2(\mathbb{R}^3)$
2. There is a one-to-one correspondence between the two sets of eigenfcts

$$\{\varphi \in H^1(\mathbb{R}^3) : H\varphi = -e\varphi\}$$
$$\{\phi \in L^2(\mathbb{R}^3) : K_e\phi = \phi\}$$

3. There is a one-to-one correspondence between the eigenvalues, in the sense that $N_e = B_e$ for

$N_e = \#\{\text{eigenvalue of } H \text{ less than or equal to } -e\}$

$B_e = \#\{\text{eigenvalues of } K_e \text{ greater than or equal to } 1\}$ $\square$

Now:
$$\sum_{j\geq 0} |E_j| = \sum_{j\geq 0} e_j = \int_0^\infty N_e \, de \overset{(3.)}{=} B_e \leq \sum_{\lambda_j \geq 1} \lambda_j(e)^2 \leq \mathrm{Tr}(K_e^2)$$

$\uparrow$ eigenvalues of K_e

$$\mathrm{Tr}(K_e^2) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} V_-(x)\, G_e(x-y)^2\, V_-(y)\, dx\,dy \underset{CS}{\leq} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} V_-(x)^2 G_e(x-y)^2$$

$$= \int_{\mathbb{R}^3} V_-(x)^2 dx \int_{\mathbb{R}^3} G_e(y)^2 dy = \int_{\mathbb{R}^3} V_-(x)^2 \underbrace{\int_{\mathbb{R}^3} \frac{dk}{(|2\pi k|^2 + e)^2}}_{= c\, e^{-\frac{1}{2}}}$$

this still needs to be integrated from 0 to ∞ in e! Problem!

Trick:

$$W_e(x) := (V(x) + e/2)_- = \max(-V(x) - e/2, 0) \leq V_-(x)$$

By min-max principle:

$$N_e \leq N_e(-V_-) = N_{\frac{e}{2}}\left(-V_- + \frac{e}{2}\right) \leq N_{\frac{e}{2}}(-W_e).$$

Then:

$$\sum_{j \geq 0} |E_j| \leq \int_0^\infty de \; N_{\frac{e}{2}}(-W_e) \leq C \int_0^\infty \left(\frac{e}{2}\right)^{-\frac{1}{2}} \int_{\mathbb{R}^3} W_e(x)^2 \, dx \, de$$

$$= C \int_0^\infty \left(\frac{e}{2}\right)^{-\frac{1}{2}} \int_{\mathbb{R}^3} \left(V(x) + \frac{e}{2}\right)_-^2 \, dx \, de$$

$$= C \int_0^\infty \left(\frac{e}{2}\right)^{-\frac{1}{2}} \int_{\mathbb{R}^3} \left(V_-(x) - \frac{e}{2}\right)_+^2 \, dx \, de$$

$$= C \int_{\mathbb{R}^3} \int_0^{2V_-(x)} \left(\frac{e}{2}\right)^{-\frac{1}{2}} \left(V_-(x) - \frac{e}{2}\right)^2 \, dx \, de$$

and the final computation as before.